Physics for Mascians

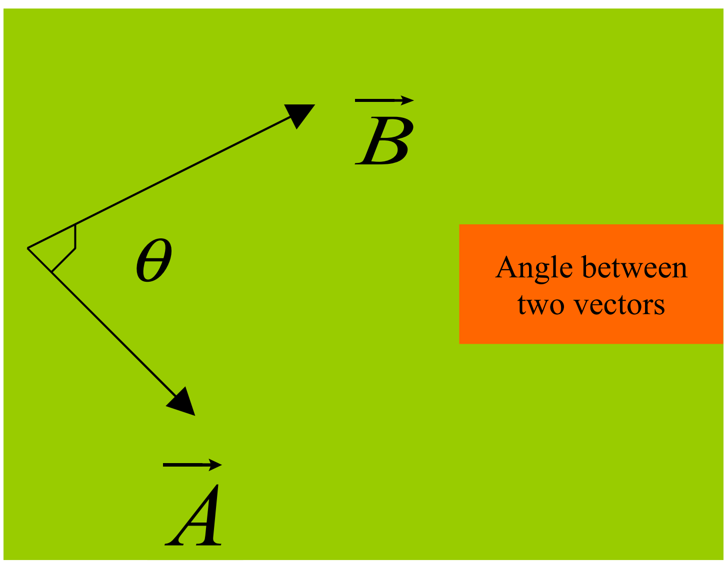
Vector Multiplication

Two kinds of vector multiplication are of interest. The first is the scalar or dot product, and as the name implies, the result is a scalar. This product is denoted by $\vec{A} \bullet \vec{B}$ and is defined as

$$\vec{A} \bullet \vec{B} = AB\cos\theta$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$ and is illustrated in Figure 1: Notice that $A\cos\theta$ is the projection of the length of the arrow representing $\vec{A}$ onto the direction of $\vec{B}$ and similarly, $B\cos\theta$ is the projection of the length of the arrow of $\vec{B}$ onto the direction of $\vec{A}$.

Figure 1:



If $\vec{A}$ and $\vec{B}$ are perpendicular ($\theta = 90^o$), then $\vec{A} \bullet \vec{B} = 0$, and such vectors are said to be *orthogonal.* Also $\vec{A} \bullet \vec{A} = A^2$ gives the square of the magnitude of $\vec{A}$.

The other kind of product, is the vector or cross product denoted by $\vec{A} \times \vec{B}$ and as the name implies the product is itself a vector. By definition if

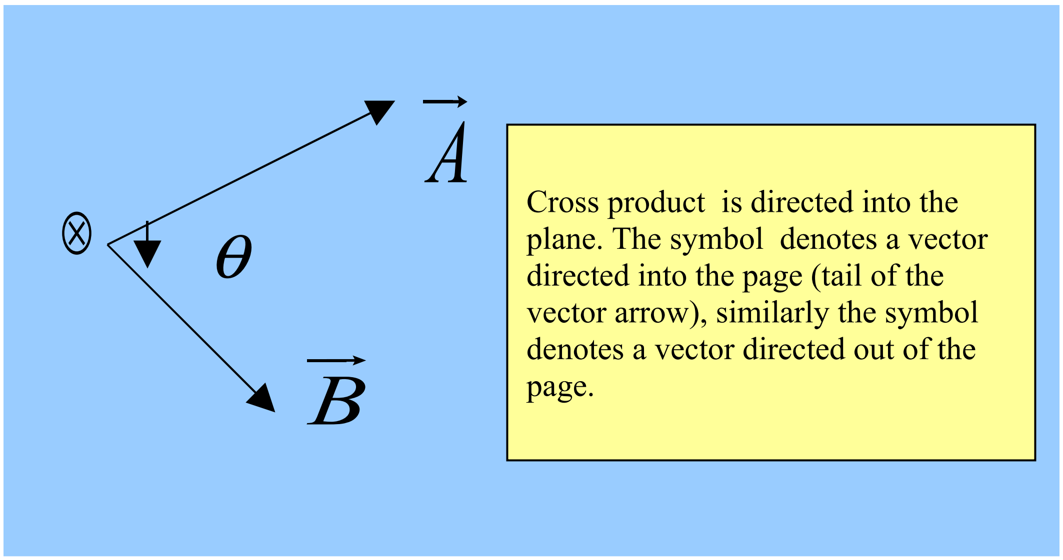
$$\vec{C} = \vec{A} \times \vec{B}$$

then

$$C = AB\sin\theta$$

And the direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ (hence to the plane they define) with its sense along this perpendicular that of a right hand screw when $\vec{A}$ is rotated into $\vec{B}$. This is illustrated in Figure 2. Notice that if $\vec{A}$ and $\vec{B}$ are parallel $\vec{A} \times \vec{B} = 0$

Figure 2:

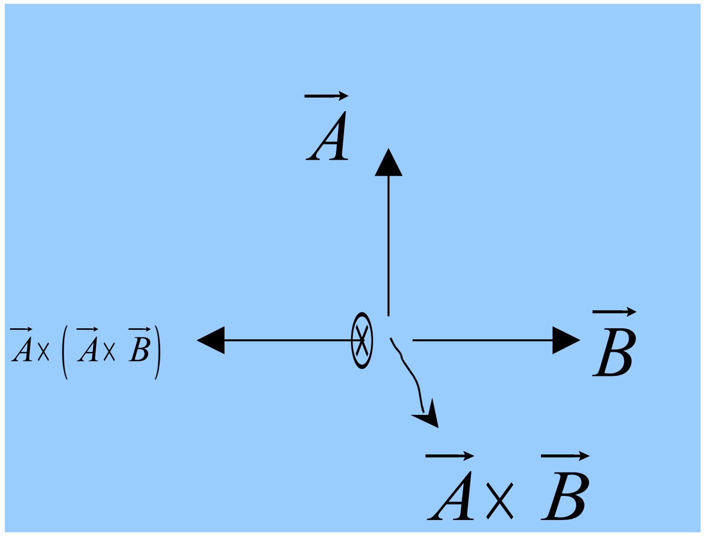


Some useful properties are that

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$
$$\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$$

Each of which follows from the definitions. Only the vector product is defined for three or more vectors. . For example $\vec{A} \bullet \vec{B} \bullet \vec{C}$ is meaningless, and $\vec{A} \times \vec{B} \times \vec{C}$ is defined only if one specifies by brackets which cross product is to be done first. For example, let us suppose that $\vec{A}$ and $\vec{B}$ have unit length each and are perpendicular. Then $\vec{A} \times \left(\vec{B} \times \vec{A}\right) = -\vec{B}$ may be seen from Figure 3, but $\left(\vec{A} \times \vec{A}\right) \times \vec{B} = 0$

Figure 3:



Unit Vectors

If $\vec{A}$ is a vector, then $\vec{a}=\vec{A}/A$ is a vector of unit magnitude whose direction is the same as that of $\vec{A}$. This is called unit vector. A unit vector does not carry physical units.

Any two non-parallel vectors (say $\vec{A}$ and $\vec{B}$) determine a plane and any vector in that plane can be written in terms of $\vec{A}$ and $\vec{B}$ as

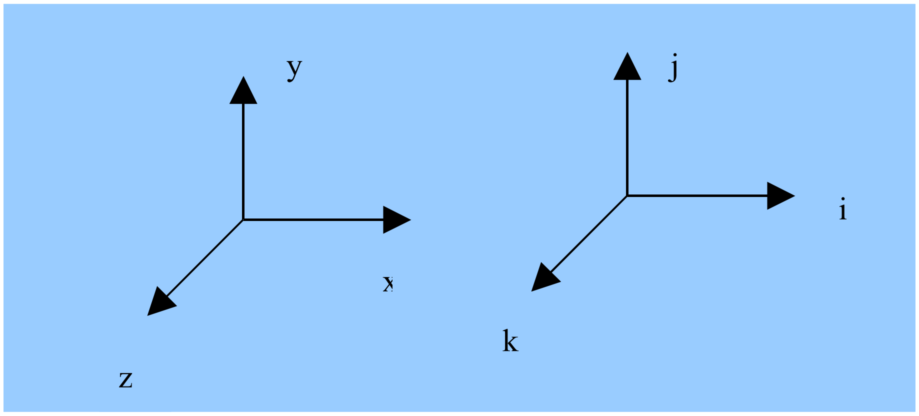
$$\vec{C}=a\vec{A}+b\vec{B}$$

Similarly in three dimensions, our common experience space, any vector can be written in terms of three non-coplanar vectors. The three constitute a coordinate system. Unit vectors are of use here.

One of the most useful coordinate systems is the Cartesian system with unit vectors along the x, y and z axes as shown in Figure 4. For these unit vectors, the special symbols $\vec{i},\vec{j}$ and $\vec{k}$ are used and they have very beautiful properties.

$$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1 \quad ; \quad \vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{k}=0$$
$$\vec{i}\times\vec{i}=\vec{j}\times\vec{j}=\vec{k}\times\vec{k}=0$$
$$\vec{i}\times\vec{j}=\vec{k},\vec{j}\times\vec{k}=\vec{i},\vec{k}\times\vec{i}=\vec{j}$$

Figure 4:



Any vector $\vec{F}$ in three dimensions may be written as

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

The scalar product of two vectors may be written in terms of unit vectors.

Let $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$, then

$$\begin{aligned}\vec{A} \bullet \vec{B} &= \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right) \bullet \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Similarly, the vector product of $\vec{A}$ and $\vec{B}$ may also be expressed in terms of unit vectors

$$\begin{aligned}\vec{A} \times \vec{B} &= \left(A_x\vec{i} + A_y\vec{j} + A_z\vec{k}\right) \times \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right) \\ &= \left(A_yB_z - A_zB_y\right)\vec{i} + \left(A_zB_x - A_xB_z\right)\vec{j} + \left(A_xB_y - A_yB_x\right)\vec{k}\end{aligned}$$ (Why?)

The equation above is “neatly” described when it is expressed in determinant form

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The vector product follows the distributive property of multiplication

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)$$

Example

Given two vectors $\vec{A} = 4\vec{i}+3\vec{j}$ and $\vec{B} = -2\vec{i}+6\vec{j}$, find:

a. $\vec{A}\bullet\vec{B}$
b. the magnitude of $\vec{A}$
c. $\vec{A}\times\vec{B}$
d. a unit vector in the direction of $\vec{B}$
e. $\vec{A}\times\left(\vec{A}\times\vec{B}\right)$
f. $\vec{A}\times\left(\vec{A}\times\vec{B}\right)$ in terms of $\vec{A}$ and $\vec{B}$

Solution:

a.
$$\vec{A}\bullet\vec{B} = A_xB_x + A_yB_y$$
$$\vec{A}\bullet\vec{B} = \left(4\vec{i}+3\vec{j}\right)\bullet\left(-2\vec{i}+6\vec{j}\right) = (4)(-2)+(3)(6) = 10$$

b.
$$\vec{A}\bullet\vec{A} = A^2 = \left(4\vec{i}+3\vec{j}\right)\bullet\left(4\vec{i}+3\vec{j}\right) = 16+9 = 25$$
$$A = \sqrt{A^2} = 5$$

Note that the positive root is considered.

c.
$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 4 & 3 & 0 \\ -2 & 6 & 0 \end{vmatrix} = \left[3(0)-6(0)\right]\vec{i}+\left[4(0)-(-2)(0)\right]\vec{j}+\left[4(6)-(-2)(3)\right]\vec{k}$$
$$\vec{A}\times\vec{B} = 0+0+30\vec{k} = 30\vec{k}$$

d. First determine the magnitude of $\vec{B}$

$$B^2 = \vec{B} \bullet \vec{B} = 4 + 36 = 40 \Rightarrow \ B = \sqrt{40}$$

Let b be the unit vector parallel to $\vec{B}$, then

$$\vec{b} = \frac{\vec{B}}{B} = \frac{-2\vec{i} + 6\vec{j}}{\sqrt{40}} = -\frac{-2}{\sqrt{40}}\vec{i} + \frac{6}{\sqrt{40}}\vec{j}$$

e. $\vec{A} \times \left(\vec{A} \times \vec{B}\right) = \left(4\vec{i} + 3\vec{j}\right) \times 30\vec{k} = (120)\left(\vec{i} \times \vec{k}\right) + 90\left(\vec{j} \times \vec{k}\right) = -120\vec{j} + 90\vec{i}$

f. $\vec{A} \times \left(\vec{A} \times \vec{B}\right) = a\vec{A} + b\vec{B} = a\left(4\vec{i} + 3\vec{j}\right) + b\left(-2\vec{i} + 6\vec{j}\right)$, where a and b are to be determined. Thus we have

$$4a\vec{i} + 3a\vec{j} - 2b\vec{i} + 6b\vec{j} = (4a - 2b)\vec{i} + (3a + 6b)\vec{j} = -120\vec{j} + 90\vec{i}$$

Because $\vec{i}$ and $\vec{j}$ are not parallel the components may be set equal and we have

$$4a - 2b = 90 \ \text{(coefficients of } \vec{i}\text{)}$$
$$3a + 6b = -120 \ \text{(coefficients of } \vec{j}\text{)}$$

From which we have a = 10 and b= -25. Thus, $\vec{A} \times \left(\vec{A} \times \vec{B}\right) = 10\vec{A} - 25\vec{B}$

PHYSICS APPLICATIONS

Work. Consider a constant force $\vec{F}$ acting o a body and displacing the body through $\vec{d}$. The work done by the force is given by:

$$W = \vec{F} \bullet \vec{d} = Fd\cos\theta$$

Find the work done in moving an object along a vector $\vec{d} = 2\vec{i} - 3\vec{j} + \vec{k}$ if the applied force is $\vec{F} = 4\vec{i} + \vec{j} - 2\vec{k}$. Assume SI units for both quantities.

Solution:

$$W = \vec{F} \bullet \vec{d} = \left(4\vec{i} + \vec{j} - 2\vec{k}\right) \bullet \left(2\vec{i} - 3\vec{j} + \vec{k}\right)$$

$$W = 4(2) + 1(-3) + (-2)(1) = 8 - 3 - 2 = 3J$$

A force $\vec{F} = 3\vec{i} - 4\vec{j} + \vec{k}$ is applied at the point (2, -1, 2). Find the torque about point (1, 3, -2).

Solution:

$$\tau = \vec{r} \times \vec{F}$$

First we find $\vec{r}$ which is the radius vector from point (1, 3, -2) to point (2, -1, 2).

$$\vec{r} = (2-1)\vec{i} + (-1-3)\vec{j} + (2+2)\vec{k} = \vec{i} - 4\vec{j} + 4\vec{k}$$

Thus

$$\tau = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 1 & -4 & 4 \\ 3 & -4 & 1 \end{vmatrix} = \left[-4(1) - (-4)(4)\right]\vec{i} + \left[1(1) - 3(4)\right]\vec{j} + \left[1(-4) - (3)(-4)\right]\vec{k}$$

$$\tau = \vec{r} \times \vec{F} = 12\vec{i} + 11\vec{j} + 8\vec{k}$$

In the product $\vec{F} = q\vec{v} \times \vec{B}$, take q = 2; $\vec{v} = 2\vec{i} + 4\vec{j} + 6\vec{k}$ and $\vec{F} = 4\vec{i} - 20\vec{j} + 12\vec{k}$, what then is $\vec{B}$ in unit vector notation if $B_x = B_y$

Solution:

Let $B_x = B_y = C$, then using the definition of the cross product,

$$4\vec{i} - 20\vec{j} + 12\vec{k} = \begin{vmatrix} i & j & k \\ 4 & 8 & 12 \\ C & C & B_z \end{vmatrix} = \left[8B_z - 12C\right]\vec{i} + \left[4B_z - 12C\right]\vec{j} + \left[4C - 8C\right]\vec{k}$$

$$\begin{aligned} 12 &= -4C \\ C &= -3 \end{aligned} \quad \text{(coefficients of } \vec{k}\text{)}$$

$$\begin{aligned} 4 &= 8B_z - 12C \\ 4 &= 8B_z + 36 \quad \text{(coefficients of } \vec{i}\text{)} \\ B_z &= -4 \end{aligned}$$

Thus

$$\overrightarrow{B} = B_x i + B_y j + B_z k = -3i - 3j - 4k$$