SCALING

Scaling is used in science to determine the effect of size on function. Can we live normal, active lives if reduced to the size of an ant or if increased to the size of King Kong? *Scaling is also used to construct models of larger or smaller objects.* For example, a geologist can build a model whose changes in a short time represent structural changes in the earth that take millions of years to occur. Engineers and architects use scaling to make models for testing designs of buildings, airplane wings, power plants and so forth.

To start our discussion of scaling, consider two groups of blocks. The first group consists of two cubes one sitting on the other. Each cube has sides of length 1 unit. The two blocks together have dimensions 1 x 1 x 2.

The second group of blocks has the same shape as the first but is three times longer in each direction. Its dimensions are 3 x 3 x 6. The scaling factor of the second group of blocks compared to the first is 3 since each dimension of the second group is three times longer than the similar dimension of the first group.

The scaling factor λ is the ratio of a dimension d' of one object and the similar dimension d of another object whose shape is the same as the first.

$$\lambda = \frac{d'}{d}$$

If the objects have similar shapes, any dimension can be used to calculate the scaling factor, just as long as the same dimension is used for each object.

How does the area of a similarly shaped objects depend on the scaling factor? Consider the cross section area of the first group of blocks described earlier. The cross-section area A = 1 x 1 = 1, whereas for the larger set of blocks, the cross-section are A' = 3 x 3 = 9. The ratio of the two areas A'/A = 9/1 = 9 is the square of the scaling factor. The ratio of the areas of the side surfaces of the blocks also equals the scaling factor squared since , for the side surfaces A'/A = (3 x 6)/(1 x 2) = 0.

Areas scale as the square of the scaling factor

$$\frac{A'}{A} = \lambda^2$$

This seems reasonable. Each dimension has increased by λ, and the area is the product of two dimension.

How do the volumes of these two similarly shaped groups of blocks compare? The volume of the small group of blocks is 1 x 1 x 2 = 2, whereas that of the larger groups is 3 x 3 x 6 = 54. The ratio of their volumes V'/V = 54/2 = 27, is the cube of the scaling factor ($3^3 = 27$). This also seems reasonable. Each dimension has increased by λ, and the volume is the product of three dimensions. Thus, the volume of one object should be λ^3 times that of the other object.

Volume scale as the cube of the scaling factor:

$$\frac{V'}{V} = \lambda^3$$

If two objects have equal and uniform density, then their masses and weights also scale in proportion to the cube of the scaling factor, because mass and weight are proportional to volume. If the volume of the object is λ^3 bigger than that of another, its mass is also λ^3 bigger assuming that their densities are equal.

Example:

During a one-year period, a young girl grows so that each dimension in her body increases by 5 percent. By what percentage does her weight increase, assuming that her density remains unchanged?

Solution

One of the dimensions of her body, such as her height, increases in one year from a value d to a value d', where, for a 5 percent increase d' = 1.05 d; the scaling factor is 1.05. Her weight increases in proportion to the scaling factor cubed, or

$$\frac{W'}{W} = \lambda^3 = (1.05)^3 = 1.16$$

Thus her weight increases by 16 percent.

Example:

(King Kong's Achilles tendon) A giant such as the popular movie figure King Kong would have many structural problems if shaped like a man or woman. Suppose the giant is 60 ft tall and stands on the balls of his feet with heels slightly elevated. Will his Achilles tendon tear? The tension in the Achilles tendon of a 6-ft tall, 180-lb man of similar shape and stance is 310 lbs. The man's tendon tears if the tension exceeds 1500 lbs.

Solution:

The giant is 10 times taller than the man; thus λ = 10. The giant's mass and weight are λ^3 = 10^3 =1000 times greater than that of the man. However, the strength of the giant's muscles and tendons is only λ^2, or 100 times greater than that of the man, since strength depends on the cross sectional area of muscles, tendons and bones. Thus, the giant can exert 100 times more force with his muscles and tendons than can the man and can support 100 times more force on his bones. But, unfortunately, he is 1000 times heavier. The tension in the Achilles tendon of a 180-lb man when his heels are slightly elevated is about 310 lb. For the giant, the tension is one thousand times greater (since it is 1000 times heavier), that is, 1000 x 310 lb = 310, 00 lbs.

A man's Achilles tendon will reap when the tension exceeds about 1500 lb. Since the giant's tendon is 100 times larger in cross section, it will rip if the tension exceeds λ^2(1500 lb) = 150,000 lb. But to stand with his heel slightly off the ground requires a tension of 310,000 lb. Merely standing on the ball of his foot will rip the tendon, If the giant runs or jumps, the tension is even greater. The compression force on his bones and joints could cause fractures when standing. A 60 ft giant shaped like a man would be unable to stand, walk or run.

Example:

Parachutists have survived impacts in snowbanks even though their parachutes did not open. Their speed at impact was about 50 m/s (120 mi/h), the terminal speed of a falling skydiver. Will a giant skydiver 10 times the length of an averaged-sized woman (λ =10) or an elf one-tenth the length of an average-seized woman (λ =0.1) have a greater terminal speed as they reach the snowbank?

Solution:

At terminal speed, the weight of the falling person is balanced by the opposing drag force of the air R..

$$W = R = \frac{1}{2} C\rho\, A v_T^2$$

Solving for the terminal velocity

$$v_T = \sqrt{\frac{2W}{C\rho\, A}}$$

Where C is the drag coefficient (the same roughly for all three divers), ρ is the density of the air, A is the cross-sectional area of the diver.

The ratio of the terminal speed of a giant or elf and that of a woman with similar shape is

$$\frac{v_T'}{v_T} = \sqrt{\frac{2W'/C\rho\, A'}{2W/C\rho\, A}} = \sqrt{\frac{W'}{W}\frac{A}{A'}} = \sqrt{\lambda^3 \frac{1}{\lambda^2}} = \sqrt{\lambda}$$

Note that the weight scales as the third power of the scaling factor and the area scales as the square of the scaling factor. We see that the ratio of the terminal speeds of similarly shaped falling objects scales as the square root of the scaling factor. Using v_{Twoman}= 50 m/s, we find that

$$v'_{T_{giant}} = \sqrt{10}x50m/s = 160m/s$$

Whereas

$$v'_{T_{elf}} = \sqrt{0.1}x50m/s = 16m/s$$

The elf moves at a much slower speed when arriving at the snowbank. An extension of calculations such as this indicates that the force per unit area needed to stop the giant is over 100 times greater than that needed to stop the elf, if they stop in the same distance. (why?).

Gravity is a less important factor for small animals than for large ones. An insect can hang easily from a ceiling, but a Giant such as King Kong will break bones and rip tendons while walking because of the huge mass that must be supported by his legs. It appears that gravity limits the size of animals living on earth. Had life evolved on a less massive planet, animals might have been larger.

Geologists have used scaling to construct models of geological formations. The materials for these models are chosen so that their change with time represents similar structural changes expected in the earth. The time scale can be reduced so that changes in the model occurring in several hours represent similar changes in the earth that may occur in millions of years. To build these models, the scaling of distance, time and mass must be considered carefully, as must the effect of these choices of scale on other quantities such as density, force, pressure and flow rate. The following example illustrate the effect of scale on a derived quantity such as flow rate.

Example:
A reservoir fills with water at a rate of 1000 m^3/min. A model of the reservoir is built so that each dimension in the model is 10^{-3} times the similar dimension of the reservoir. At what rate should water flow into the model?

Solution

The scaling factor for distance and for time are

$$\text{Distance scaling factor} = \lambda = \frac{d_{model}}{d_{reservoir}} = 10^{-3}$$

$$\text{Time scaling factor} = \tau = \frac{t_{model}}{t_{reservoir}} = 10^{-2}$$

Where t_{model} represents the time between events happening in the model (such as the start and completion of filling the model with water). And $t_{reservoir}$ represents the time for the same events in the actual reservoir.

The flow rate of water into the model Q_{model} is the volume V_{model} of water that flows into the model divided by the time t_{model} required for this flow:

$$Q_{model} = \frac{V_{model}}{t_{model}}$$

Since volume scales as the cube of the distance scaling factor, we find that

$$Q_{model} = \frac{V_{model}}{t_{model}} = \frac{\lambda^3 V_{reservoir}}{\tau\, t_{reservoir}} = \frac{\lambda^3}{\tau} Q_{reservoir}$$

$$Q_{model} = \frac{\left(10^{-3}\right)^3}{\left(10^{-2}\right)}\left(1000m^3/\min\right) = 10^{-4}\, m^3/\min$$