QUADRATIC FORMULA

The solutions for the general quadratic equation $ax^2 + bx + c = 0 \quad a \neq 0$, can be solved using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is best that you memorize this formula and keep in mind the meaning of the roots and the significance of the $\pm$ sign. The expression $\left(b^2 - 4ac\right)$ under the radical sign is called the discriminant.

If $\left(b^2 - 4ac\right)$ is positive, there are two real and unequal roots

If $\left(b^2 - 4ac\right)$ is zero, the two roots are real and equal

If $\left(b^2 - 4ac\right)$ is negative, the roots are imaginary

An additional property is that the sum of the roots of the quadratic equation is equal to $\frac{b}{a}$ and the products of the roots is equal to $\frac{c}{a}$, which may be used to check the correctness of the roots.

Oftentimes in physics we encounter the problem of interpreting the physical meaning of the roots. It is possible that both roots, or values of x, are acceptable. Sometimes, however, one of the roots has no physical meaning and must be discarded. Let us give some examples of how the roots of the quadratic equation are interpreted.

Example 1:

When a body is projected directly upward with an initial velocity v_o, the height reached after t seconds is

$$h = v_o t - \frac{1}{2} g t^2$$

If the initial velocity with which a body is projected upward is 49 m/s, how long will it take before the body reaches a height of 78.4 m?

Solution:

Rearranging the terms in the equation above, we have

$$4.9t^2 - v_o t + h = 0 \Rightarrow 4.9t^2 - 49t + 78.4 = 0$$

This is a quadratic equation where a = 4.9, b = -49 and c = 78.4. Solving for t

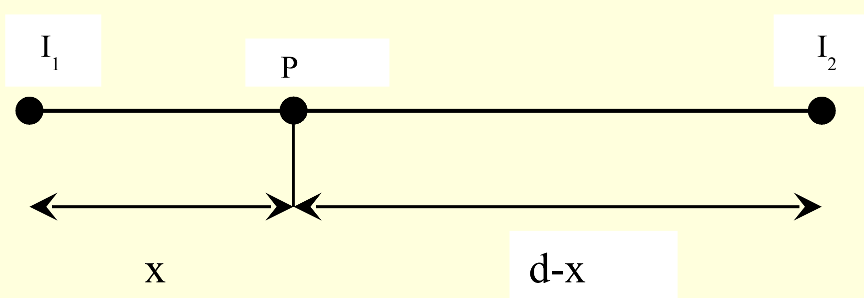
$$t = \frac{-(49) \pm \sqrt{(-49)^2 - (4)(4.9)(78.4)}}{2(4.9)}$$

$$t = 2s \quad and \quad t = 8s$$

Example 2:

The brightness E on a surface which is a distance d from a point source of intensity I candle power is I/d^2.

Problem: Given two point sources of light I_1=100 cp and I_2= 400 cp, which are at a distance of 3 meters apart. Where between the two sources should a screen be placed so that it will be equally illuminated by the two sources.



Let P be the point where the screen is placed so it will be equally illuminated by the two sources, and let x be the distance of P from I_1. Equating the two expressions for intensity of illumination, we have

$$\frac{I_1}{x^2} = \frac{I_2}{(d - x)^2}$$

After substituting values,

$$\frac{100}{x^2} = \frac{400}{(3 - x)^2}$$

Cross multiplying and simplifying,

$$9-6x+x^2=4x^2 \Rightarrow 3x^2+6x-9=0$$

Applying the quadratic formula,

$$x=\frac{-6\pm\sqrt{36+108}}{6}$$
$$x=1m, \quad x=-3m$$

The negative value of x stands for P to the left of I_1. This answer has to be discarded because the problem specifies the point to be somewhere between the two sources. At x = -3m, however, the screen will also be equally illuminated by the two sources.

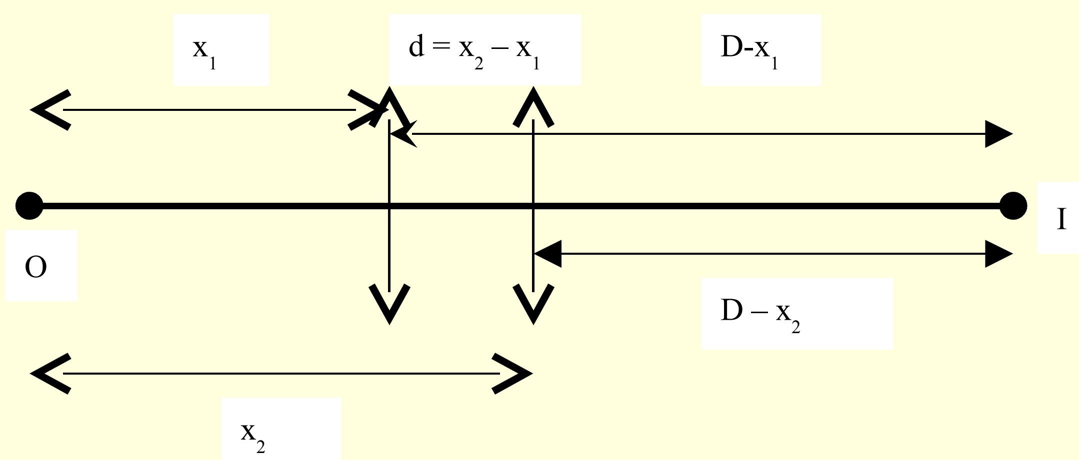
Example 3: A luminous object and a screen are a fixed distance D apart. (1) Show that a converging lens of focal length, f, placed between object and screen, will form a real image of a screen for two lens positions that are separated by a distance

$$d=\sqrt{D\left(D-4f\right)}$$

2. Show that

$$\left(\frac{D-d}{D+d}\right)^2$$

Gives the ratio of the two image sizes for these two positions of the lens.



Solution:

Using the thin lens equation

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$$

If x is the distance of the object from the lens and D is the distance between the object and the image, the thin lens equation gives

$$\frac{1}{x}+\frac{1}{D-x}=\frac{1}{f}$$

Solving for x

$$\frac{1}{x}+\frac{1}{D-x}=\frac{1}{f}$$

$$\frac{D-x+x}{x\left(D-x\right)}=\frac{1}{f}$$

$$Df=Dx-x^2$$

$$x^2-Dx+Df=0$$

$$x=\frac{D\pm\sqrt{D^2-4Df}}{2}$$

$$d=x_2-x_1=\frac{D+\sqrt{D^2-4Df}}{2}-\frac{D-\sqrt{D^2-4Df}}{2}$$

$$d=\frac{2\sqrt{D^2-4Df}}{2}=\sqrt{D\left(D-4f\right)}$$

2. The magnification of the image is given by

$$M=\frac{S_1}{S_0}=-\frac{q}{p}$$

For the first image:

$$M_1=\frac{D-x_1}{x_1}=\frac{D-\left(\frac{D-d}{2}\right)}{\frac{D-d}{2}}=\frac{\frac{2D-D+d}{2}}{\frac{D-d}{2}}=\frac{D+d}{D-d}$$

For the second image:

$$M_2 = \frac{D - x_2}{x_2} = \frac{D - \left(\frac{D+d}{2}\right)}{\frac{D+d}{2}} = \frac{\frac{2D - D - d}{2}}{\frac{D+d}{2}} = \frac{D-d}{D+d}$$

Thus

$$\frac{M_2}{M_1} = \frac{S_2/S_0}{S_1/S_0} = \frac{S_2}{S_1} = \frac{\frac{D-d}{D+d}}{\frac{D+d}{D-d}} = \left(\frac{D-d}{D+d}\right)^2$$

Example 4:

An stone is thrown height h with an initial velocity v_o that makes an angle θ with the horizontal. How long will it take the stone to reach the ground? How far from the base of the cliff did it hit the ground?

Solution:

Using the equation for motion with constant acceleration, we have:

$$y = y_0 + v_{oy}t - \frac{1}{2}gt^2$$

Placing the origin at the base of the cliff and taking the upward direction as positive

$$0 = h + v_o \sin\theta\, t - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 - v_o \sin\theta\, t - h = 0$$

This is a quadratic equation with $a = \frac{1}{2}g, \quad b = -v_o \sin\theta, \quad c = -h$. Solving for t using the quadratic formula,

$$0 = h + v_o \sin\theta \, t - \frac{1}{2} g t^2$$

$$a = \frac{1}{2} g, \quad b = -v_o \sin\theta, \quad c = -h$$

$$t = \frac{-\left(-v_o \sin\theta\right) \pm \sqrt{\left(-v_o \sin\theta\right)^2 - 4\left(\frac{1}{2} g\right)\left(-h\right)}}{2\left(\frac{1}{2} g\right)}$$

$$t = \frac{v_o \sin\theta + \sqrt{\left(v_o \sin\theta\right)^2 + 2gh}}{g}$$

Note that if h = 0 (corresponding to y = y_o= 0), the time reduces to:

$$t = \frac{2 v_o \sin\theta}{g}$$

2) The range is given by

$$R = v_{ox} t = v_o \cos\theta \, t$$

Substituting eq 1 to the equation for the range gives

$$R = v_{ox} t = \left(v_o \cos\theta\right)\left(\frac{v_o \sin\theta + \sqrt{\left(v_o \sin\theta\right)^2 + 2gh}}{g}\right)$$

Note again that if h = 0

$$R = \left(v_o \cos\theta\right)\left(\frac{v_o \sin\theta + \sqrt{\left(v_o \sin\theta\right)^2}}{g}\right) = \frac{2 v_o^2 \sin\theta \, \cos\theta}{g} = \frac{v_o^2 \sin 2\theta}{g}$$

But if h is small (but not equal to zero), what will be the range? This can be answered using the binomial approximation

$$\left(1 + x\right)^n \approx 1 + nx$$

First. Simplify the equation for the time.

$$t = \left(\frac{v_o \sin\theta + \sqrt{(v_o \sin\theta)^2 + 2gh}}{g} \right) = \frac{v_o \sin\theta}{g} \left(1 + \sqrt{1 + \left(\frac{2gh}{(v_o \sin\theta)^2} \right)} \right)$$

But since h is small,

$$\left(\frac{2gh}{(v_o \sin\theta)^2} \right) << 1$$

And that

$$\sqrt{1 + \left(\frac{2gh}{(v_o \sin\theta)^2} \right)} \approx 1 + \frac{1}{2} \left(\frac{2gh}{(v_o \sin\theta)^2} \right) = 1 + \left(\frac{gh}{(v_o \sin\theta)^2} \right)$$

The time therefore is given by

$$t = \frac{2v_o \sin\theta}{g} + \frac{h}{v_o \sin\theta}$$

And the range becomes:

$$R = \frac{v_o^2 \sin 2\theta}{g} + \frac{h}{\tan\theta}$$