PHYSICS FOR MASCIANS

ARITHMETIC AND GEOMETRIC PROGRESSIONS

An ***arithmetic progression*** is a linear function which is a sequence in which each term, after the first, is obtained from the preceding term by adding to it the same fixed number known as the common difference.

Let

a_1 be the first term
d, be the common difference,
n, the number of terms and
a_n the nth term.

Then the second term is

$$a_2 = a_1 + d$$
$$a_3 = a_1 + 2d$$
$$a_4 = a_1 + 3d$$
.
.
.
$$a_n = a_1 + (n - 1)d$$

The sum of the first n terms of an arithmetic progression is

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Example 1:

A freely-falling body drops through a height of $h = \frac{1}{2}gt^2$

During the first second, h_1 = 4.9 meters

During the 2nd second, $h_2 - h_1 = 4.9x4 - 4.9 = 14.7m$

During the 3rd second, $h_3 - h_2 = 4.9x9 - 4.9x4 = 24.5m$

During the 4th second, $h_4 - h_3 = 4.9x16 - 4.9x9 = 34.3m$

It is clear that we have a sequence with a common difference d = 9.8 m

$$14.7m - 4.9m = 9.8m$$
$$24.5m - 14.7m = 9.8m$$
$$34.3m - 24.5m = 9.8m$$

This arithmetic progression has the nth term as

$$h_n = 4.9 + 9.8(n - 1)$$
$$h_n = (9.8n - 4.9)$$

Problem:

1. How far does the body fall during the 10th second?
2. How far does it fall during the first 10 seconds?

Solution:

1. Following the sequence, during the 10th second,

   $h_{10} - h_9 = (4.9x10^2) - (4.9x9^2) = 490 - 396.9 = 93.1m$ or applying the equation for arithmetic progression, we have

   $a_{10} = a_1 + 9.8(10 - 1) = 4.9 + (9.8x9) = 93.1m$

2. $h_{10} = \frac{1}{2}gt^2 = 4.9(100) = 490m$ or applying the equation for the sum of nth terms,

$$S_{10} = \frac{10(4.9 + 93.1)}{2} = 490m$$

A ***geometric progression,*** an exponential function is a sequence in which each term, after the first, is obtained by multiplying the preceding term by a fixed number known as the common ratio.

If a is the the number and r is the common ratio, then the nth term a_n is given by

$$a_n = ar^{n-1}$$

The sum of the first n terms of a geometric progression is

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)}$$

One interesting example of a geometric progression is the rebound of an elastic ball dropped from a certain height. It usually rebounds to a constant fractional part of the preceding height of fall.

Example 2:

A rubber ball dropped from a height of four meters rebounds to a height ½ of the previous distance it has fallen.

1. How far does it rise on the 6th rebound?
2. What is the total distance travelled at the top of the 6th rebound?
3. What is the total distance travelled?

Solution: For the rebounds this is a geometric progression with the first term 2 and common ratio ½. The nth term is $2\left(\frac{1}{2}\right)^{n-1}$

Answer:

1. $a_6 = ar^5 = 2\left(\frac{1}{2}\right)^5 = \frac{1}{16}m$

2. $$S_6(\text{for rebounds}) = \frac{a\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = \frac{2\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = 3\frac{15}{16}m$$

$$S_6(\text{for drops}) = \frac{a\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = \frac{4\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = 7\frac{14}{16}m$$

The total distance at the top of the sixth rebound is $3\frac{15}{16} + 7\frac{14}{16} = 11\frac{13}{16}m$

4. For total distance, n is equal to infinity or $S_\infty = \frac{a}{1-r}$

$$S(\text{for the } \infty \text{ drops}) = \frac{4m}{1-\frac{1}{2}} = 8m$$

$$S(\text{for the } \infty \text{ rebounds}) = \frac{2m}{1-\frac{1}{2}} = 4m$$

Total distance = 12 m.

We note that the remaining distance after six rebounds is only 3/16 meter showing the slow decrease in the remaining distances.

Radioactive elements decay at a constant rate which depends on the element. The half life is defined as the time it takes ½ of the atoms in the sample to disintegrate. If the half life of an element is 20 days, then a 1-gram sample of the element is reduced to ½ gram after 20 days. This ½ gram is reduced to 1/2 of 1/2 , or ¼ g, after another 20 days; this ¼ gram is reduced to 1/8 gram after another 20days, and so on. This is then a geometric series where the constant ratio is ½.

Example 3:

The half life of a certain radioisotope is 28 years. How long will it take a 1-kg sample of this radioisotope to be reduced to 1/128 kg due to radioactive disintegration?

Solution: The decrease goes like this

Time t	Years	0	28	56	84	…..
Amount	Kg	1	½	¼	1/8	….

Here, the first term a, is 1 and the nth term is 1/128,

$$a_n = ar^{n-1}$$

$$\frac{1}{128} = 1\left(\frac{1}{2}\right)^{n-1}$$

Taking the logarithms of both sides:

$$\ln\left(\frac{1}{128}\right) = \ln\left(\frac{1}{\ }\right)^{n-1}$$

$$\ln 1 - \ln 128 = (n-1)\ln\frac{1}{2}$$

$$n = 8$$

There are therefore 7 half lives, and 7 x 28 years = 196 years.