Physics for Mascians

EXPONENTIAL FUNCTION

We live in a world where growth has become a way of life: Our population grows by a small percentage each year, as does our use of energy, the amount of rice consumed, the area of the earth covered by asphalt and concrete, the cost of one pan de sal, the national debt and the money in our savings account (assuming there are no withdrawals). If the growth is steady, these quantities can be described by the exponential function. In this article we examine some of the fundamental properties of this function and its application in Physics.

GROWTH RATE ΔN/Δt

Exponential growth (or decay) occurs when the rate of change of a quantity is proportional to the quantity itself. Suppose that you have some money in a savings account. The money ΔN that is added to your account during a time Δt depends on three quantities: (1) the amount N of money already in your account, (2) the interest rate being paid, and (3) the time Δt that the money is collecting interest. These quantities are related by the equation

$$\Delta N = kN\Delta t$$

$$\frac{\Delta N}{\Delta t} = kN, \text{ (equation 1)}$$

where k is a proportionality constant that depends on the interest rate being paid and $\Delta N/\Delta t$ is called the growth rate.

The meanings of the proportionality constant k becomes apparent when we rearrange the equation above:

$$k = \frac{\Delta N/N}{\Delta t}$$

We see that k is the fractional change in N – that is $\Delta N/N$ --per time Δt. If the money in the savings account increases by 6 percent per year, then the fractional change in the money during the year is $\Delta N/N = 0.06$ and the proportionality constant

$$k = \frac{\Delta N/N}{\Delta t} = \frac{0.06}{1yr} = 0.06yr^{-1}$$

Note that k has dimensions of time^{-1}.
When dealing with quantities that "grow" with time , it is common to indicate their percentage growth rate P. For a money in a savings account, P = 6 percent/ yr

We relate the percentage growth rate P to k by multiplying by 100 that is

$$P = \text{Percentage growth rate} = 100k$$

THE EXPONENTIAL FUNCTION

Equation 1, can also be expressed in exponential form. Using calculus, let us derive an equation that can be used to calculate the value of N at some future time t if its value N_o is known at time zero.

Rearranging equation 1:

$$\frac{\Delta N}{N} = k\Delta t \quad (2)$$

During an infinitesimal interval of time, dt, ΔN = dN, thus (2) can be written as

$$\frac{dN}{N} = kdt \quad (3)$$

Integrating both sides of equation (3),

$$\int \frac{dN}{N} = \int kdt = k\int dt = kt + C_1 \quad (4)$$

Substituting $\int \frac{dN}{N} = \ln N$ in (4),

$$\ln N = kt + C_1 \quad (5)$$

In exponential form, (5) may be written as

$$N = e^{kt+C_1} = \left(e^{kt}\right)\left(e^{C_1}\right) = C_2 e^{kt}$$

Where $C_2 = e^{C_1}$

Substituting $t = 0$,

$$N = C_2 e^{k(0)} = C_2(1) = C_2$$

But when the time $t = 0$, $N = N_o$, thus

$$N = N_o e^{kt} \quad (6)$$

If k is a positive number,, exponential growth occurs. If k is a negative number (or if a negative sign appears in front of a positive-valued k) exponential decay occurs. The concept of exponential growth (and decay) can be used to describe the decay of radioactive elements. It can also de used to describe the "decay" of the intensity level of sound (and radiation) with distance r.

DOUBLING TIME

Equation 6 can be used to derive a simple expression for calculating the time needed for a quantity's value to double. Rearranging (6) and taking the natural logarithm of each side, we find that

$$t = \frac{\ln\left(N/N_o\right)}{k} \quad (7)$$

The growing quantity has a value N_o at time zero and increases to a value $N = 2N_o$ in a time T, called the doubling time, given by the expression

$$T = \frac{\ln\left(2N_o/N_o\right)}{k} = \frac{\ln 2}{k} = \frac{0.693}{k} \quad (8)$$

If we now substitute k = P/100 into the above equation, we find

$$T = \frac{69.3}{P} \approx \frac{70}{P} \quad (9)$$

Where P is the percentage growth rate of the quantity. For example, if the world's population growth rate is 1.9 percent/yr, then the population doubling time is

$$T = \frac{70}{1.9 yr^{-1}} = 37 yr$$

If the world's yearly energy use is increasing at a rate of 4 percent/yr, the energy used per year will double in

$$T = \frac{70}{4 yr^{-1}} = 18 yr$$

The units need not be years. For example, if a colony of bacteria grows at a rate of 3.5 percent/min, then the doubling time is

$$T = \frac{70}{3.5\min^{-1}} = 20\min$$

The value of an exponentially increasing quantity at some time in the future can be determined using this idea of doubling time. Note, for example, that in one doubling time the original value N_o of a quantity doubles to become $2N_o$. In the next doubling time, the quantity increases by another factor of 2 to become $2(2N_o) = 2^2N_o = 4 N_o$. After a third doubling time, the quantity's value becomes $2 (2^2N_o) = 2^3N_o= 8N_o$. In general, in a time $t = nT$, where n is the number of doubling times, the value of a quantity increases to

$$N = 2^n N_o \text{ in a time } t = nT \quad (10)$$

where N_o was its original value at the start of that time period. Suppose, for example, that the cost of medical care increases at a rate of 14 percent. By how many would the cost increase in 20 years? Using eq. (9), we find that the doubling time is $T \approx \frac{70}{P} \approx \frac{70}{14yr^{-1}} = 5years$. The number of doubling times in 20 years is $n = \frac{t}{T} = \frac{20yr}{5yr} = 4$. Thus the total cost should increase by a factor $\frac{N}{N_o} = 2^4 = 16.$ Equation (10) can be used even if n is not an integer.

HALF LIFE

Equations (6) and (7) can also be used to derive a simple expression for calculating the time needed for a quantity's value to be reduced in half. Recall that

$$t = \frac{\ln\left(N/N_o\right)}{k}$$

The "decaying" quantity has a value N_o at time zero and reduces to a value $N = \frac{1}{2}N_o$ in a time $T_{1/2}$, called the half life, given by the expression

$$T_{1/2} = \frac{\ln\left(\frac{1}{2}N_o \Big/ N_o\right)}{k} = \frac{\ln\left(N_o/2N_o\right)}{k} = \frac{\ln\left(1/2\right)}{k} = \frac{-0.693}{k} \quad (11)$$

We can also derive an equation that can be used to calculate the value of N at some future time t, given its half life $T_{1/2}$. Rearranging Eq. 11

$$k = \frac{\ln\left(1/2\right)}{T_{1/2}} \quad (12)$$

Substituting Eq. 12 to Eq. 6, we see that

$$N = N_o e^{\left[\ln\left(1/2\right)/T_{1/2}\right]t} = N_o\left(e^{\ln\left(1/2\right)}\right)^{\frac{T_{1/2}}{t}}$$

Knowing that $e^{\ln(1/2)} = \frac{1}{2}$, the equation simplifies to

$$N = N_o\left(\frac{1}{2}\right)^{T_{1/2}/t}$$

This equation is useful in describing the decay of a radioactive substance.